

Pole structure and biharmonic fields in conformal QFT in four dimensions*

Nikolay M. Nikolov¹,
Karl-Henning Rehren^{1,2},
Ivan Todorov¹

¹ Institute for Nuclear Research and Nuclear Energy,
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria;
mitov@inrne.bas.bg, toдоров@inrne.bas.bg

² Institut für Theoretische Physik, Universität Göttingen,
Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany;
rehren@theorie.physik.uni-goe.de

Abstract

Imposing Huygens' Principle in a 4D Wightman QFT puts strong constraints on its algebraic and analytic structure. These are best understood in terms of “biharmonic fields”, whose properties reflect the presence of infinitely many conserved tensor currents. In particular, a universal third-order partial differential equation is derived for the most singular parts of connected scalar correlation functions. This PDE gives rise to novel restrictions on the pole structure of higher correlation functions. An example of a six-point function is presented that cannot arise from free fields. This example is exploited to study the locality properties of biharmonic fields.

1 Introduction

In spite of the quantitative successes of renormalized perturbation theory, no non-trivial quantum field theory (QFT) in four spacetime dimensions (4D) has been constructed rigorously. It is sometimes suggested that the Wightman axioms might be too restrictive, but most attempts at relaxing them lead to physically unacceptable consequences. E.g., a violation of causality cannot be exponentially small [14]. Admitting indefinite (physical) Hilbert spaces not only jeopardizes the statistical interpretation of correlations, but it makes statements about convergence of approximations delicate if not meaningless.

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On the contrary, the usual attitude is to strengthen the axioms, e.g., by imposing additional symmetries or phase space or additivity properties. This facilitates the analysis of models but clearly reduces the number of theories. Even worse, demanding too much may trivialize the theory. E.g., requiring that the two-point function be supported on a mass shell [9] or only to have finite mass support [5], already forces the field to be a (generalized) free field.

It is one of the merits of the axiomatic approach [18, 8] that it allows to pinpoint such obstructions even before a model is formulated. It makes no assumptions about intermediate steps and limits through which a theory is constructed. Referring only to its intrinsic features, it avoids assigning significance to the artifacts of the description. Especially, it guides navigation between Scylla (physical meaningless theories) and Charybdis (free fields) in the quest for a proper set of model assumptions that can possibly be satisfied by other than free fields.

In a series of recent papers [12, 13], two of us have suggested to demand “global conformal invariance” (GCI) which is the postulate that the conformal group is represented in a true (i.e., not a covering) representation. (The term “global” stresses the fact that the conformal transformations are defined globally on the Dirac compactification of Minkowski spacetime.)

GCI implies rationality of all correlation functions [13]. On one side, this is a desired feature since it allows to parameterize each correlation function by the coefficients of a finite set of admitted rational structures. The latter are determined by conformal invariance, and by the unitarity bound on the representations of the conformal group that possibly contribute to operator product expansions (OPE) giving rise to upper bounds on the pole in each pair of variables. The coefficients are then further restricted only by Hilbert space positivity. While the latter is highly nontrivial to control in general, there has been considerable progress at the four-point level [7, 10].

On the other side, GCI is equivalent to Huygens locality [10], i.e., commutativity of fields not only at spacelike but also at timelike separation. This feature seems to be conspicuously close to free field theory, since any scattering of field quanta should give rise to causal propagation within the forward lightcone. Indeed, it has been shown [4] that if a Huygens local QFT has a complete particle interpretation, then its scattering matrix is trivial. Therefore, GCI allows only nontrivial theories without asymptotic completeness, which may not be entirely unphysical in a scale invariant theory. Note that by rationality, all scaling dimensions are integer numbers, which does not mean that they are canonical. It is conceivable that in a QFT with anomalous dimensions, at some value of the coupling constant all dimensions simultaneously become integers. Examples for such a hazard are well known in 2D conformal QFT [6].

We shall report here on a recent analysis [11] of the consequences of the following fact. For a symmetric tensor field of rank r and dimension d , one calls $d - r$ the

“twist”. Twist two fields are necessarily conserved, because their two-point functions, completely determined by r and d , are conserved. (Hilbert space positivity crucially enters in this argument: if the norm square of a vector vanishes, then the vector itself vanishes. The rest of the argument invokes the Reeh-Schlieder theorem for which in turn locality and energy positivity are essential.) These conservation laws can be reformulated in the form

$$\square_x V(x, y) = 0 = \square_y V(x, y) \quad (1.1)$$

where $V(x, y)$ is the sum of all twist two contributions to the OPE $\phi_1(x)\phi_2(y)$ of two scalar fields of equal dimension d . We say that the “bi-field” V is biharmonic.

Our first main result is the unravelling of a hidden consequence of this equation: a third-order partial differential equation to be satisfied by (certain parts of) all correlation functions involving $\phi_1(x)\phi_2(y)$, that is a necessary and sufficient condition for biharmonicity.

We present a solution to this PDE, and the corresponding transcendental six-point correlation function of $V(x, y)$ that cannot be produced by Wick products of free fields. On the basis of this solution, which we believe to be prototypical for the general case, we then study the locality properties of the bi-field.

2 Biharmonicity

We first explain how harmonicity of $V(x, y)$ serves to define its correlation functions. Because $V(x, y)$ is biharmonic, there are in fact two such prescriptions, and hence $V(x, y)$ is overdetermined. The consistency condition gives rise to a restriction on the correlation functions involving $\phi_1(x)\phi_2(y)$.

The crucial fact is the following “classical” result [3]. If $u(x)$ is a power series in $x \in \mathbb{R}^D$ or \mathbb{C}^D , then there is a unique “harmonic decomposition”

$$u(x) = v(x) + x^2 \hat{u}(x), \quad (2.1)$$

such that v is harmonic ($\square_x v = 0$) and \hat{u} is again a power series. We call v the “harmonic part” of u . (Questions of convergence will be discussed in Sect. 5.)

We apply this fact as follows. Twist two contributions to correlations involving $\phi_1(x)\phi_2(y)$ have a leading singularity $((x - y)^2)^{1-d}$, while all higher twist contributions are less singular. Therefore, the (Huygens bilocal, but not conformally covariant) bi-field $U(x, y)$ defined by

$$\phi_1(x)\phi_2(y) - \langle \phi_1(x)\phi_2(y) \rangle =: ((x - y)^2)^{-(d-1)} \cdot U(x, y) \quad (2.2)$$

is regular at $(x - y)^2 = 0$. $U(x, y)$ contains all twist \geq two contributions to the OPE, and because the twist $>$ two contributions are suppressed by another factor $(x - y)^2$, we may write

$$U(x, y) = V(x, y) + (x - y)^2 \hat{U}(x, y) \quad (2.3)$$

where both V and \hat{U} are regular at $(x - y)^2 = 0$. Now consider any correlation function

$$u(x, y, \dots) = \langle U(x, y) \phi_3(x_3) \cdots \phi_n(x_n) \rangle, \quad (2.4)$$

where \dots stands for the arguments of all other fields. Its Taylor expansion in x around y is a power series in $x - y$ with coefficients independent of x . Thus, by (2.3) and because $\square_x V(x, y) = 0$, the desired correlation function (again as a power series)

$$v(x, y, \dots) = \langle V(x, y) \phi_3(x_3) \cdots \phi_n(x_n) \rangle \quad (2.5)$$

is the harmonic part of this series. By construction, $v(x, y, \dots)$ transforms like the correlation function of a conformal bi-scalar of dimension $(1, 1)$.

On the other hand, the Taylor expansion of $u(x, y, \dots)$ in y around x is another power series in $x - y$, whose coefficients do not depend on y , and because also $\square_y V(x, y) = 0$, $v(x, y, \dots)$ may as well be determined as the harmonic part of this latter series.

This overdetermination imposes a consistency condition on the function $u(x, y, \dots)$. Its nontriviality may be seen from the following example. Consider $u(x, y, \dots) = (y - x_6)^2 / (x - x_3)^2 (y - x_4)^2 (y - x_5)^2$. This function is harmonic with respect to x , hence its harmonic part with the first prescription coincides with u itself, but it is not harmonic with respect to y , so the harmonic part with respect to the second prescription differs from u , and the two definitions of v are conflicting each other. Thus, a function $u(x, y, \dots)$ as in this example cannot occur as a correlation function of $U(x, y)$.

We conclude that biharmonicity of the bi-field $V(x, y)$, which follows from the conservation of conformal twist two tensor fields, implies a nontrivial restriction on the possible correlation functions $u(x, y, \dots)$ of the bi-field $U(x, y)$, and hence on the correlations involving $\phi_1(x) \phi_2(y)$.

We shall now turn this condition into a partial differential equation.

Global conformal invariance implies that scalar correlation functions are Laurent polynomials in the variables $\rho_{ij} = \rho_{ji} = (x_i - x_j)^2$ of the form

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \sum_{\underline{\mu}} C_{\underline{\mu}} \prod_{i < j} \rho_{ij}^{\mu_{ij}} \quad (2.6)$$

where the integer powers $\mu_{ij} = \mu_{ji}$ satisfy the homogeneity rules

$$\sum_j \mu_{ij} = -d_i, \quad (2.7)$$

and the absence of non-unitary representations of the conformal group in the OPE implies the lower bound for the connected parts of (2.6)

$$2\mu_{ij} \geq -d_i - d_j. \quad (2.8)$$

Let ϕ_1 and ϕ_2 have the same dimension d . It follows that all correlations (2.4) (that give contributions to (2.6)) are Laurent polynomials in ρ_{ij} , which are separately homogeneous of total degree -1 in ρ_{1k} ($k \neq 1$) and in ρ_{2k} ($k \neq 2$), and which are true polynomials in ρ_{12} . Because all terms involving a factor ρ_{12} have zero harmonic part in the harmonic decompositions, we need to consider only the function u_0 which is the contribution of order $(\rho_{12})^0$ of u . Then u_0 is separately homogeneous of total degree -1 in ρ_{1k} ($k > 2$) and in ρ_{2k} ($k > 2$).

It is now important that the harmonic part v is a real analytic function in a neighborhood of $x_1 = x_2$, provided $(x_2 - x_i)^2 \neq 0$ for all $j > 2$ (see Sect. 5). We may therefore expand v as a power series $\sum_{n=1}^{\infty} h_n/n! \cdot \rho_{12}^n$. The coefficients h_n are functions of all the remaining variables $\rho_{ij} \neq \rho_{12}$, and are separately homogeneous of total degree $-n-1$ in ρ_{1k} ($k > 2$) and in ρ_{2k} ($k > 2$).

Let us write $\partial_{jk} = \partial_{kj} = \frac{\partial}{\partial \rho_{jk}}$. Then the wave operator \square_{x_1} has the form

$$\square_{x_1} = -4 \left(\sum_{2 \leq j < k \leq n} \rho_{jk} \partial_{1j} \partial_{1k} \right) = -4(D_1 + E_1 \partial_{12}) \quad (2.9)$$

valid on homogeneous functions of total degree -1 in ρ_{1k} ($k \neq 1$) [10], where

$$D_1 = \sum_{3 \leq j < k \leq n} \rho_{jk} \partial_{1j} \partial_{1k} \quad \text{and} \quad E_1 = \sum_{3 \leq i} \rho_{2i} \partial_{1i}. \quad (2.10)$$

Similarly, replacing $1 \leftrightarrow 2$ everywhere, one represents $\square_{x_2} = -4(D_2 + E_2 \partial_{12})$. Thus, the two conditions $\square_{x_1} f = 0 = \square_{x_2} f$ give rise to two recursive systems of partial differential equations for the coefficient functions h_n of the form

$$E_1 h_{n+1} = -D_1 h_n \quad \text{and} \quad E_2 h_{n+1} = -D_2 h_n. \quad (2.11)$$

At $n = 0$ we obtain the integrability condition

$$(E_1 D_2 - E_2 D_1) h_0 = (E_2 E_1 - E_1 E_2) h_1. \quad (2.12)$$

Because h_1 is separately homogeneous of total degree -2 in ρ_{1k} ($k \geq 3$) and in ρ_{2k} ($k \geq 3$), the commutator $(E_2 E_1 - E_1 E_2)$ vanishes on h_1 . Since v is the harmonic part of u , its leading term h_0 equals u_0 . Thus, we arrive at

Result 1: *The function u_0 solves the third-order partial differential equation*

$$(E_1 D_2 - E_2 D_1) u_0 = 0. \quad (2.13)$$

Next, when (2.13) holds and the recursion is solved for h_1 , one has $(D_1 E_2 - D_2 E_1) h_1 = -(D_1 D_2 - D_2 D_1) h_0 = 0$ because D_1 and D_2 commute. But $D_1 E_2 - D_2 E_1 = E_2 D_1 - E_1 D_2$. Therefore, the integrability condition for the next step of the recursion is automatically satisfied, and the argument passes to all the higher steps. Thus, (2.13) secures solvability of the entire recursive systems (2.11).

3 Consequences

One should worry how there can be a new universal (model-independent) partial differential equation for the correlation functions. It is important to notice that this PDE can not be regarded as some “equation of motion”. The reason is that it is satisfied only by the leading part u_0 of u . The splitting of u into $u_0 +$ a remainder does not correspond to any local decomposition of the bi-field $U(x, y)$. Thus, the PDE (2.13) cannot be formulated as a differential equation for some (bi-)fields in the theory.

Instead, it should be understood as a kinematical constraint. Because its solutions u_0 must at the same time be Laurent polynomials, the PDE rather selects a (finite) set of admissible singularity structures, that depends on the dimensions of the scalar fields involved through the lower bounds on μ_{ij} .

Indeed, we have shown in [11] that the PDE (2.13) implies the following constraint on the pole structure of a Laurent polynomial u_0 in ρ_{1k} and ρ_{2k} ($k > 2$), that is homogeneous of degree -1 in both sets of variables separately: Suppose u_0 contains a monomial

$$\prod_{k>2} \rho_{1k}^{\mu_{1k}} \rho_{2k}^{\mu_{2k}} \times \text{other factors} \quad (3.1)$$

where the other factors depend only on ρ_{kl} ($k, l > 2$). If there are $i \neq j$ such that $\mu_{1i} < 0$ and $\mu_{1j} < 0$ (a “double pole” in x_1), then one must have $\mu_{2k} \geq 0$ for all $k > 2$, $k \neq i, j$. In particular, this excludes “triple poles”, because a triple pole in x_1 would imply that *all* $\mu_{2k} \geq 0$, contradicting homogeneity. The most involved possible pole structure of u_0 is therefore of the form

$$\frac{\text{polynomial}}{\rho_{1i}^p \rho_{1j}^q \rho_{2i}^r \rho_{2j}^s} \times \text{other factors} \quad (3.2)$$

where the polynomial takes care of the proper homogeneity. The corresponding contribution to the connected correlations involving $\phi_1(x)\phi_2(y)$ is therefore

$$\frac{1}{\rho_{12}^{d-1}} \frac{\text{polynomial}}{\rho_{1i}^p \rho_{1j}^q \rho_{2i}^r \rho_{2j}^s} \times \text{other factors.} \quad (3.3)$$

Note that no constraints arise on higher twist contributions ($\mu_{12} > 1 - d$) or on poles of two fields with $d_1 \neq d_2$.

The interest in double poles is due to the fact that twist two bi-fields made of free fields, such as $:\varphi(x)\varphi(y):$ or $(x - y)^\mu : \bar{\psi}(x) \gamma_\mu \psi(y) :$ are always Wick bilinears, so that their correlation functions can never contain a double pole. A nontrivial double pole solution (an example will be displayed below) is therefore a candidate for a Huygens local QFT not generated by free fields.

We have also shown in [11] that

Result 2: A correlation function involving $V(x_1, x_2)$, i.e., the harmonic part of the Laurent polynomial $u_0(x_1, x_2, \dots)$, is again a Laurent polynomial if and only if u_0 does not contain a double pole in x_1 or x_2 .

Four-point functions $\langle U(x_1, x_2)\phi_3(x_3)\phi_4(x_4) \rangle$ can never exhibit double poles in x_1 or x_2 , just “by lack of independent variables”. Therefore, four-point functions of twist two bi-fields are always rational. From this one can deduce that their partial wave expansion cannot terminate after finitely many terms, i.e., the OPE of $\phi_1(x)\phi_2(y)$ must contain infinitely many conserved tensor fields.

If all fields are scalars of dimension 2, hence $\mu_{ij} \geq -1$, double poles cannot occur in any n -point function subject to the cluster decay property. We have exploited this fact in [11] to prove that scalar fields ϕ of dimension 2 are always Wick products of the form $\sum M_{ij}:\varphi_i\varphi_j:(x)$ of massless free fields. In this argument, Hilbert space positivity plays a crucial role because one has to solve a moment problem in order to get the correct coefficients for all n -point functions simultaneously. (When we do not insist that the theory possesses a stress-energy tensor with a finite two-point function, then the fields ϕ may also have contributions of generalized free fields.)

The simple pole structure of correlation functions of dimension 2 fields can be converted into commutation relations of the twist two biharmonic fields occurring in their OPE. The result is an infinite-dimensional Lie algebra, whose unitary positive-energy representations can be studied with methods of highest weight modules. It turns out that there are no other representations than those induced by the free field construction [2].

4 An example with double poles

The following six-point structure solves the PDE (2.13) both in the variables x_1, x_2 and in the variables x_5, x_6 :

$$u(x_1, \dots, x_6) = \frac{(\rho_{15}\rho_{26}\rho_{34} - 2\rho_{15}\rho_{23}\rho_{46} - 2\rho_{15}\rho_{24}\rho_{36})_{[1,2][5,6]}}{\rho_{13}\rho_{14}\rho_{23}\rho_{24} \cdot \rho_{34}^{d'-3} \cdot \rho_{35}\rho_{45}\rho_{36}\rho_{46}}, \quad (4.1)$$

where $(\dots)_{[i,j]}$ stands for the antisymmetrization in the arguments x_i, x_j , and $\rho_{ij} = (x_i - x_j)^2$ as before. This structure in addition obeys all homogeneity rules (2.7), pole bounds (2.8) and cluster conditions in order to qualify as (a contribution to) the correlation function

$$\langle U(x_1, x_2)\phi'(x_3)\phi'(x_4)U(x_5, x_6) \rangle \quad (4.2)$$

where the scalar field ϕ' has dimension d' . The multiple poles in the variables x_3, x_4 do not contradict the previous argument (Sect. 3) excluding triple poles in the twist two “channel”, when either d (the dimension of the fields ϕ_1, ϕ_2 in (2.2), generating

U) or d' is > 2 , because they don't arise in a channel of twist two ($1/\rho_{34}^{d'-3}$ is twist six, and $1/\rho_{3i}$ and $1/\rho_{4i}$ are twist two only if $d = d' = 2$).

We determined the corresponding (contribution to the) correlation

$$\langle V(x_1, x_2) \phi'(x_3) \phi'(x_4) V(x_5, x_6) \rangle, \quad (4.3)$$

$v(x_1, \dots, x_6)$, as the (simultaneous) harmonic part(s) of $u(x_1, \dots, x_6)$. Let

$$s = \frac{\rho_{12}\rho_{34}}{\rho_{13}\rho_{24}}, \quad t = \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{24}} \quad (4.4)$$

denote the conformal cross ratios, and s' and t' the same with 1, 2 replaced by 5, 6. Then

$$\begin{aligned} v(x_1, \dots, x_6) &= u(x_1, \dots, x_6) \cdot g(t, s) g(t', s') + \\ &+ \frac{2(\rho_{13}\rho_{24} \cdot \rho_{35}\rho_{46})_{[1,2][5,6]}}{\rho_{13}\rho_{14}\rho_{23}\rho_{24} \cdot \rho_{34}^{d'-2} \cdot \rho_{35}\rho_{45}\rho_{36}\rho_{46}} \cdot (1 - g(t, s) g(t', s')) \end{aligned} \quad (4.5)$$

has the required power series expansion $u(x_1, \dots, x_6) + O(\rho_{12}, \rho_{56})$ provided $g(s, t)$ is of the form $g(s, t) = \sum_{n \geq 0} g_n(t)/n! \cdot s^n$ with $g_0(t) = 1$, and it is harmonic in all four variables x_1, x_2, x_5, x_6 provided g solves the PDE

$$\left((1 - t\partial_t)(1 + t\partial_t + s\partial_s) - [(1 - t\partial_t) + t(2 + t\partial_t + s\partial_s)]\partial_s \right) g = 0. \quad (4.6)$$

The solution is

$$\begin{aligned} g(s, t) &= \frac{1}{s} \cdot \left[Li_2(u) + Li_2(v) - Li_2(u + v - uv) \right] + \\ &+ \frac{t}{s} \cdot \left[Li_2\left(\frac{-u}{1-u}\right) + Li_2\left(\frac{-v}{1-v}\right) - Li_2\left(\frac{uv - u - v}{(1-u)(1-v)}\right) \right], \end{aligned} \quad (4.7)$$

where u and v (apologies for the duplicate use of letters!) here stand for the ‘‘chiral’’ variables defined by the algebraic equations

$$s = uv \quad \text{and} \quad t = (1 - u)(1 - v). \quad (4.8)$$

Li_2 is the dilogarithmic function defined by analytic continuation of its integral or power series representations ($0 \leq x < 1$)

$$Li_2(x) = - \int_0^x \frac{\log(1-t)}{t} dt = \sum_{n>0} \frac{x^n}{n^2}. \quad (4.9)$$

Notice that g is regular at $s = 0$ in spite of the prefactors $\sim 1/s$. This transcendental correlation function can definitely not be produced by free fields. It was found by turning the differential equation (4.6) into the recursive system

$$(1 + (n+1)t - t(1-t)\partial_t)g_n = (1 - t\partial_t)(n + t\partial_t)g_{n-1} \quad (4.10)$$

with $g_0(t) = 1$, and resumming the solution

$$\frac{g_n(t)}{n!} = \frac{n!(n+1)!}{(2n+1)!} \cdot {}_2F_1(n, n+1; 2n+2; 1-t) \quad (4.11)$$

by exploiting the integral representation of hypergeometric functions.

5 Local commutativity

We shall now discuss the issue of local commutativity of the bi-field $V(x, y)$. The naive argument would go as follows: since $U(x, y)$ is Huygens bilocal in the sense of local commutativity for spacelike or timelike separation from x and y , the correlation functions

$$u_k(x, y, \dots) = \langle \phi_3(x_3) \dots \phi_k(x_k) U(x, y) \phi_{k+1}(x_{k+1}) \dots \phi_n(x_n) \rangle \quad (5.1)$$

are independent of the position k where $U(x, y)$ is inserted. By the uniqueness of the harmonic decomposition, the same should be true for their harmonic parts

$$v_k(x, y, \dots) = \langle \phi_3(x_3) \dots \phi_k(x_k) V(x, y) \phi_{k+1}(x_{k+1}) \dots \phi_n(x_n) \rangle, \quad (5.2)$$

hence $V(x, y)$ commutes with $\phi_k(x_k)$.

However, this argument is not correct because of convergence problems of the power series. The transcendentality of the correlation function (4.5) shows that $V(x, y)$ in this case is certainly not a Huygens bilocal field, which must have rational correlation functions by the same argument [13] as for Huygens local fields. On the other hand, the Result 2 in Sect. 3 yields a necessary and sufficient condition (obviously violated by (4.1)):

Result 3: $V(x, y)$ is Huygens bilocal if and only if the coefficients of the twist two pole $((x - y)^2)^{-(d-1)}$ in every correlation involving $\phi(x)\phi(y)$ (i.e., the leading parts u_0 of the correlations of $U(x, y)$) never exhibit “double poles” in the variables x or y (as explained Sect. 3).

In general, i.e., when there are double poles, $V(x, y)$ is originally only defined as a formal power series (in $x - y$) within each correlation function. Even when these series converge, it is not a priori clear what the labelling pair of points x, y has to do with its localization in the sense of local commutativity with other fields, because splitting the OPE into pieces is a highly nonlocal operation (involving projections onto eigenspaces of conformal Casimir operators).

In order to study local commutativity, we need to control convergence of the series defining the harmonic part. The latter can be addressed with the help of the “generalized residue formula”. This integral representation of the harmonic part was found recently [1] in the context of higher-dimensional vertex algebras:

$$v(x) = \frac{1}{i\pi|\mathbb{S}^{D-1}|} \int_{M_r} d^D z |_{M_r} \frac{1 - x^2/z^2}{((z - x)^2)^{D/2}} u(z). \quad (5.3)$$

Here, $z^2 = \sum_{a=1}^D z_a^2$ is the complex Euclidean square. M_r is the compact submanifold $r \cdot \mathbb{S}^1 \cdot \mathbb{S}^{D-1}$ ($\mathbb{S}^1 \subset \mathbb{C}$ is the complex unit circle, and $\mathbb{S}^{D-1} \subset \mathbb{R}^D \subset \mathbb{C}^D$ the real unit sphere), and $d^D z|_{M_r}$ the induced complex measure.

The radius $r > 0$ has to be chosen such that $u(z)$ converges absolutely for $z \in M_r$. Then for x small enough such that the kernel converges absolutely as a power series in x for every $z \in M_r$, the integral converges as a power series in x , and is independent of the choice of r .

This formula for the harmonic part w.r.t. the Euclidean Laplacian remains valid for the Lorentzian Laplacian, provided z^2 is replaced by the (complex) Lorentzian square, and the unit sphere by the set $\{(ix^0, \vec{x}) : (x^0, \vec{x}) \in \mathbb{S}^{D-1}\}$. This is true because the map $\mathbb{C}^D \rightarrow \mathbb{C}^D$, $(z^0, \vec{z}) \mapsto (iz^0, \vec{z})$, intertwines the Euclidean with the Lorentzian harmonic decomposition.

In the case at hand, where $x - y$ plays the role of x and $u(x)$ is the Taylor series around x or around y , respectively, of a Laurent polynomial with poles at $(x - x_j)^2 = 0$ and $(y - x_j)^2 = 0$, we find absolute convergence in the domain

$$\begin{aligned} \|x - y\| + \sqrt{\|x - y\|^2 + |(x - y)^2|} &< \\ &< \sqrt{\|x - x_j\|^2 + |(x - x_j)^2|} - \|x - x_j\| \quad \forall j = 1, \dots, n \end{aligned} \quad (5.4)$$

in the first case, and the same with x replaced by y in the second case. Especially, if x and y are spacelike or timelike separated from all other points x_j , these domains are not empty. We have therefore

Result 4: *The formal power series $v_k(x, y, \dots)$ for the correlation functions (5.2) converge absolutely within the domains (5.4), and the resulting functions v_k do not depend on the position k where $V(x, y)$ is inserted in (5.2).*

The issue of local commutativity of $V(x, y)$ with $\phi_k(x_k)$ now amounts to the question whether $v_{k-1} = v_k$ still holds outside the domain (5.4), as long as x_k is spacelike (or timelike) from x and y . We conservatively anticipate that the correlation functions are real analytic functions of real spacetime points within the region where local commutativity holds. Then the existence of a unique real analytic continuation from (5.4) to some other configuration implies $v_{k-1} = v_k$ at the latter configuration by virtue of Result 4, and hence commutativity. Continuation beyond a singularity requires to go through a suitable complex cone which depends on the position k where $V(x, y)$ is inserted in (5.2), hence commutativity will fail. Put differently, our strategy to establish locality by inspection of analyticity inverts the usual axiomatic reasoning [18] by which one *derives* the domain of analyticity from the *known* locality (and energy positivity).

We want to discuss specifically the local commutativity of $V(x_1, x_2)$ with $\phi'(x_3)$ in the case of the example (4.5), by studying its maximal real analytic continuation starting from the domain (5.4), which is a neighborhood of $x_1 = x_2$ where $s = 0$, $t = 1$, hence $u = v = 0$. Clearly, we can only reach configurations where $(x_1 - x_k)^2 \neq 0$

has the same sign as $(x_2 - x_k)^2$ for $k = 3$ and $k = 4$, because this is trivially true at $x_1 = x_2$ and we cannot pass through $t = 0$ or $t = \infty$ where the variables u or v in (4.7) would hit the singularities of the dilogarithmic function $Li_2(z)$ at $z = 1$ and $z = \infty$.

We claim that (4.5) has a unique real analytic continuation to *all* these points, or equivalently, that $g(s, t)$ given by (4.7) has a unique real analytic continuation in the region $t > 0$, s arbitrary (real). This is obvious for the last terms in the two lines of (4.7) because for $t > 0$ their arguments are < 1 . For the study of the remaining terms, we solve (4.8) for u and v (where it does not matter which one is which because of the manifest symmetry of (4.7) under $u \leftrightarrow v$)

$$u, v = \frac{1}{2} \left(1 - t + s \pm \sqrt{(1 - t + s)^2 - 4s} \right). \quad (5.5)$$

In the range $s \leq (1 - \sqrt{t})^2$, u and v are real and $u + v = 1 - t + s < 2$. From $(1 - u)(1 - v) = t > 0$, we see that both u and v are < 1 , and so are $\frac{-u}{1-u}$ and $\frac{-v}{1-v}$. The continuation to these points is unambiguous. In the range $(1 - \sqrt{t})^2 < s < (1 + \sqrt{t})^2$, u and v are complex and conjugate to each other, so that the first two terms in both lines of (4.7) are always the sum of the values on the two branches above and below the cut. In particular, $g(s, t)$ is real and Li_2 in (4.7) may be replaced by its real part. Finally, in the range $s \geq (1 + \sqrt{t})^2$, we find $u + v > 2$, hence both u and v and also $\frac{-u}{1-u}$ and $\frac{-v}{1-v}$ are > 1 . All four arguments hit the cut of $Li_2(z)$. But because its discontinuity is *imaginary*, the real parts are real analytic. This proves the claim.

As explained before, the maximal domain of real analyticity specifies those configurations x_1, x_2, x_3 , where $\phi'(x_3)$ commutes with $V(x_1, x_2)$. We may assume $x_4^2 \rightarrow \pm\infty$ (which can be achieved by a conformal transformation), hence $t = (x_2 - x_3)^2 / (x_1 - x_3)^2$. Thus we get commutativity whenever x_3 is *simultaneously* spacelike or timelike from x_1 and x_2 . We summarize

Result 5: *The transcendental (part of a) correlation function (4.5) is compatible with local commutativity between $V(x, y)$ and $\phi'(z)$ when $x - z$ and $y - z$ are either both spacelike or both timelike.*

The set of these configurations is locally, but *not globally* conformal invariant, since a conformal transformation may switch the sign σ of $(x - z)^2 / (y - z)^2$. This is not a contradiction: connecting configurations with $\sigma = +$ with those with $\sigma = -$ by a path in the conformal group, one must necessarily pass through $x = \infty$ or $y = \infty$, where the OPE in terms of power series ceases to make sense. This breakdown of GCI for the biharmonic field $V(x, y)$ is, of course, just another manifestation of its violation of Huygens bilocality.

It is worth noticing that another decomposition theory for the OPE in conformal QFT was developed in [16, 17]. While it is coarser than the twist decomposition (it is even trivial in the GCI case), it was found to exhibit, at least in two dimensions [15], a “localization between the points” with similar implications as the present one.

6 Conclusion

We have outlined recent progress in the intrinsic structure analysis of quantum field theories in four dimensions, under the assumption of “global conformal invariance” [11].

We have found nontrivial restrictions on the singularity structure of correlation functions. The encouraging aspect is that these restrictions allow a small “margin” beyond free correlations, for which we have given a nontrivial example. It exhibits a local but not Huygens local bi-field $V(x, y)$ whose correlation functions involve dilogarithmic functions. Local commutativity with a third field at a point z is shown to hold (in this example) whenever $x - z$ and $y - z$ are either both spacelike or both timelike. The possible failure of local commutativity when one is spacelike while the other is timelike, occurs only in correlations of at least five points, because four-point functions cannot exhibit the characteristic “double poles” in the twist two channel that are responsible for the transcendental correlations involving $V(x, y)$.

A serious question remains to be answered before our six-point structure is established as (a contribution to) a manifestly non-free correlation function: we cannot control (at the moment) Wightman positivity at the six-point level. In the case at hand, this means that we do not know whether the vectors $\phi'(z)V(x, y)|0\rangle$ span a Hilbert (sub-)space with positive metric. Because our six-point structure reduces in the leading OPE channels to five- and four-point structures that can also be obtained from free fields, positivity can only be violated in higher channels where our present knowledge of partial waves is not sufficient. Far more ambitious is the problem whether a given six-point function can be supplemented by higher correlations satisfying Wightman positivity in full generality, i.e., to recover the full Hilbert space on which the bi-field $V(x, y)$ and its generating field $\phi(x)$ act.

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